Can one test individual rationality under anonymity?

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Consider the following experiment. Suppose that there is a *large* population of (possibly heterogeneous) individuals each of whom has an order of preferences over a finite set. Some individual is randomly drawn, and is asked to choose over a subset of the finite set. The subset offered to the individual and his choice, but not his name, are recorded. Suppose that this is repeated a *large* number of times (with replacement of the individual), each time allowing the individual who is drawn to choose from a subset that need not always be the same. At the end of the experiment, we obtain a family of distributions of the choices made by the individuals, one for each one of the subsets that were being eventually offered.

Since the names of the individuals were never recorded, intuition would suggest that it is impossible to test the hypothesis that the individuals who were drawn from the population always chose rationally, in the sense of choosing the maximizer of their (fixed) orders of preferences. I now show that this intuition is incorrect, by adapting an idea of Barberá and Pattanaik (1996) to a result of McFadden and Richter (1990).

Let $X \neq \varnothing$ be the universe of choices. I assume that $\#X < \infty$. Let the population be represented by an interval $\mathcal{I}$ of measure one. Let $\mathcal{B} \subseteq \mathcal{P}(X) \setminus \{\varnothing\}$, $\mathcal{B} \neq \varnothing$, be the family of subsets of X which are eventually offered to the individuals.[1] For each $B \in \mathcal{B}$, let $\pi_B : \mathcal{P}(B) \longrightarrow [0,1]$ be the observed distribution of choices over B.

Let Γ be the set of orders over X. Clearly, $\#\Gamma = (\#X)! < \infty$. I will say that the family $\{\pi_B\}_{B \in \mathcal{B}}$ is rationalizable if there exists a function $f : \mathcal{I} \longrightarrow \Gamma$, which is measurable with respect to the Borel σ-algebra,[2] such that

$$(\forall B \in \mathcal{B})\,(\forall A \in \mathcal{P}(B)) : \pi_B(A) = \left| f^{-1}\left[\left\{\gamma \in \Gamma \mid \arg\max_B \gamma \in A\right\}\right]\right|$$

where, for $Z \subseteq \mathbb{R}$, $|Z|$ denotes Z's Lebesgue's measure. The condition means that there exists an assignment f of preferences over the population of individ-

[1] If Z is a set, I denote its power set by $\mathcal{P}(Z) = \{W \mid W \subseteq Z\}$. I endow sets with their power sets as σ-algebras out of convenience (in order to avoid measurability problems), but this is not necessary.

[2] That is, that for every $\Delta \in \mathcal{P}(\Gamma)$, $f^{-1}(\Delta)$ is a Borel set.

uals such that for each offered subset, B, and for every one of its subsets, A, the observed frequency with which the element chosen from B lies in A is the same as the "size" of the subpopulation of individuals who, under the assignment f of preferences, would choose an element of A when offered the subset B. In that sense, the assignment f rationalizes the observed distributions.

It is immediate from its definition that the hypothesis of rationalizability is refutable: under rationalizability,

$$A \subseteq B \subseteq B' \in \mathcal{B} \Longrightarrow \pi_B(A) \geqslant \pi_{B'}(A)$$

a condition which is just an application of the "regularity principle" of Block and Marshak (1960) and whose intuition is straightforward. Refutability comes from the fact that the condition is not a tautology, since the probabilities in the implication come from different distributions. The important question is whether or not this latter condition exhausts the empirical implications of the rationalizability hypothesis. The following example, which is an adaptation of one given by Barberá and Pattanaik, shows that this is not the case: the data in the example satisfy the regularity principle and yet are inconsistent with the rationalizability hypothesis.

Example 1 *Suppose that* $X = \{x, y, z\}$, $\mathcal{B} = \mathcal{P}(X) \setminus \{\emptyset\}$. *All the relevant information about* $\{\pi_B\}_{B \in \mathcal{B}}$ *is given by*

$$\begin{aligned} \pi_{\{x,y\}}(\{x\}) &= \pi_{\{y,z\}}(\{y\}) = \pi_{\{x,z\}}(\{z\}) = \frac{2}{3} \\ \pi_{\{x,y,z\}}(\{x\}) &= \pi_{\{x,y,z\}}(\{y\}) = \pi_{\{x,y,z\}}(\{z\}) = \frac{1}{3} \end{aligned}$$

Use $\langle \cdot, \cdot, \cdot \rangle$ *to explicitly denote the orders of* X *in the set* Γ. *Rationalization by* f *would require that*

$$\begin{aligned} \pi_{\{x,y\}}(\{x\}) &= \left| f^{-1}\left[\{\langle x, y, z\rangle, \langle x, z, y\rangle, \langle z, x, y\rangle\}\right]\right| = \frac{2}{3} \\ \pi_{\{x,y,z\}}(\{x\}) &= \left| f^{-1}\left[\{\langle x, y, z\rangle, \langle x, z, y\rangle\}\right]\right| = \frac{1}{3} \end{aligned}$$

from where

$$\left| f^{-1}\left[\{\langle z, x, y\rangle\}\right]\right| = \frac{1}{3}$$

Since this is symmetric, we have that for every $\gamma \in \Gamma$, $\left|f^{-1}\left[\{\gamma\}\right]\right| = \frac{1}{3}$. *But then,* $\left|f^{-1}[\Gamma]\right| = 2 > 1 = |\mathcal{I}|$ *which is impossible, since* $f : \mathcal{I} \longrightarrow \Gamma$.

I now argue that there is a stronger condition which exhausts the restrictions of the hypothesis, since it is both necessary and sufficient for rationalizability.

Define first the function $\alpha : \Gamma \times \mathcal{B} \times \mathcal{P}(X) \longrightarrow \{0, 1\}$ by

$$\alpha(\gamma, B, A) = \begin{cases} 1 & \text{if } \arg\max_B \gamma \in A \\ 0 & \text{otherwise} \end{cases}$$

Proposition 2 *The family* $\{\pi_B\}_{B\in\mathcal{B}}$ *is rationalizable if and only if for every* $K\in\mathbb{N}$ *and every sequence* $((B_k,A_k))_{k=1}^{K}$*, satisfying that for every* $k\in\{1,...,K\}$*,* $B_k\in\mathcal{B}$ *and* $A_k\in\mathcal{P}(B_k)$*, it is true that*

$$\sum_{k=1}^{K}\pi_{B_k}(A_k)\leqslant\max_{\gamma\in\Gamma}\sum_{k=1}^{K}\alpha(\gamma,B_k,A_k)$$

Proof. Necessity: Suppose that $\{\pi_B\}_{B\in\mathcal{B}}$ is rationalized by f. Define the distribution $\xi:\mathcal{P}(\Gamma)\longrightarrow[0,1]$ by

$$(\forall\Delta\in\mathcal{P}(\Gamma)):\xi(\Delta)=\left|f^{-1}[\Delta]\right|$$

which we can do since f is measurable with respect to the Borel σ-algebra, and every Borel set is Lebesgue measurable. Then,

$$(\forall B\in\mathcal{B})(\forall A\in\mathcal{P}(B)):\pi_B(A)=\xi\left(\left\{\gamma\in\Gamma\mid\arg\max_{B}\gamma\in A\right\}\right)$$

Necessity then follows from theorem 2 in McFadden and Ricther.

Sufficiency: Suppose that $\{\pi_B\}_{B\in\mathcal{B}}$ satisfies the condition. By theorem 2 in McFadden and Ricther, there exists a distribution $\xi:\mathcal{P}(\Gamma)\longrightarrow[0,1]$ such that

$$(\forall B\in\mathcal{B})(\forall A\in\mathcal{P}(B)):\pi_B(A)=\xi\left(\left\{\gamma\in\Gamma\mid\arg\max_{B}\gamma\in A\right\}\right)$$

Since Γ is finite we can enumerate its elements as

$$\Gamma=\left\{\gamma_1,\gamma_2,...,\gamma_{\#\Gamma}\right\}$$

For every $t\in\{1,2,...,\#\Gamma\}$, define $\chi_t=\sum_{\tau=1}^{t}\xi(\gamma_\tau)$. For simplicity, assume that $\mathcal{I}=[0,1]$. Then, we can define $f:\mathcal{I}\longrightarrow\Gamma$ as follows:

$$\begin{aligned}(\forall i\in[0,\chi_1)) &: f(i)=\gamma_1\\ (\forall t\in\{2,3,...,\#\Gamma-1\})\left(\forall i\in\left[\chi_{t-1},\chi_t\right)\right) &: f(i)=\gamma_t\\ \left(\forall i\in\left[\chi_{\#\Gamma-1},1\right]\right) &: f(i)=\gamma_{\#\Gamma}\end{aligned}$$

That f is measurable with respect to the Borel sets follows since for every $\Delta\in\mathcal{P}(\Gamma)$, $f^{-1}[\Delta]$ is the union of finitely many intervals. Now, let $B\in\mathcal{B}$ and $A\in\mathcal{P}(\mathcal{B})$. Then,

$$\begin{aligned}\left|f^{-1}\left[\left\{\gamma\in\Gamma\mid\arg\max_{B}\gamma\in A\right\}\right]\right| &= \sum_{t\in\{1,2,...,\#\Gamma\}\mid\arg\max_B\gamma_t\in A}\left(\chi_t-\chi_{t-1}\right)\\ &= \sum_{t\in\{1,2,...,\#\Gamma\}\mid\arg\max_B\gamma_t\in A}\xi(\gamma_t)\\ &= \xi\left(\left\{t\in\{1,2,...,\#\Gamma\}\mid\arg\max_{B}\gamma_t\in A\right\}\right)\\ &= \xi\left(\left\{\gamma\in\Gamma\mid\arg\max_{B}\gamma\in A\right\}\right)\\ &= \pi_B(A)\end{aligned}$$

where I used the σ-additivity of measure ξ. ■

The condition of the theorem, which McFadden and Richter called the Axiom of Revealed Stochastic Preferences (ARSP) is an analogous for the case of random utility of the well known Strong Axiom of Revealed Preferences. Its intuition is that events that are likely should happen often. That is, consider the following event: "for each k, when the set B_k was offered an element of A_k was chosen." Suppose that such event was likely in the sense that the left-hand side of the condition of the theorem was "high," then, it should also be true that for at least one of the preferences in Γ it often (in k) happened that the individually-rational choice over B_k lied in A_k, which made the right-hand side also "high."

ARSP is shown by McFadden and Ricther to work for finite sets X. Carvajal (2002) showed that the same condition applies for infinite sets and for collective choice problems. Necessary and sufficient conditions for rationalizability are also given for the finite case by Falmagne (1978) and by Barberá and Pattanaik.

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